ECS455: Chapter 5
OFDM

5.4 Cyclic Prefix (CP)

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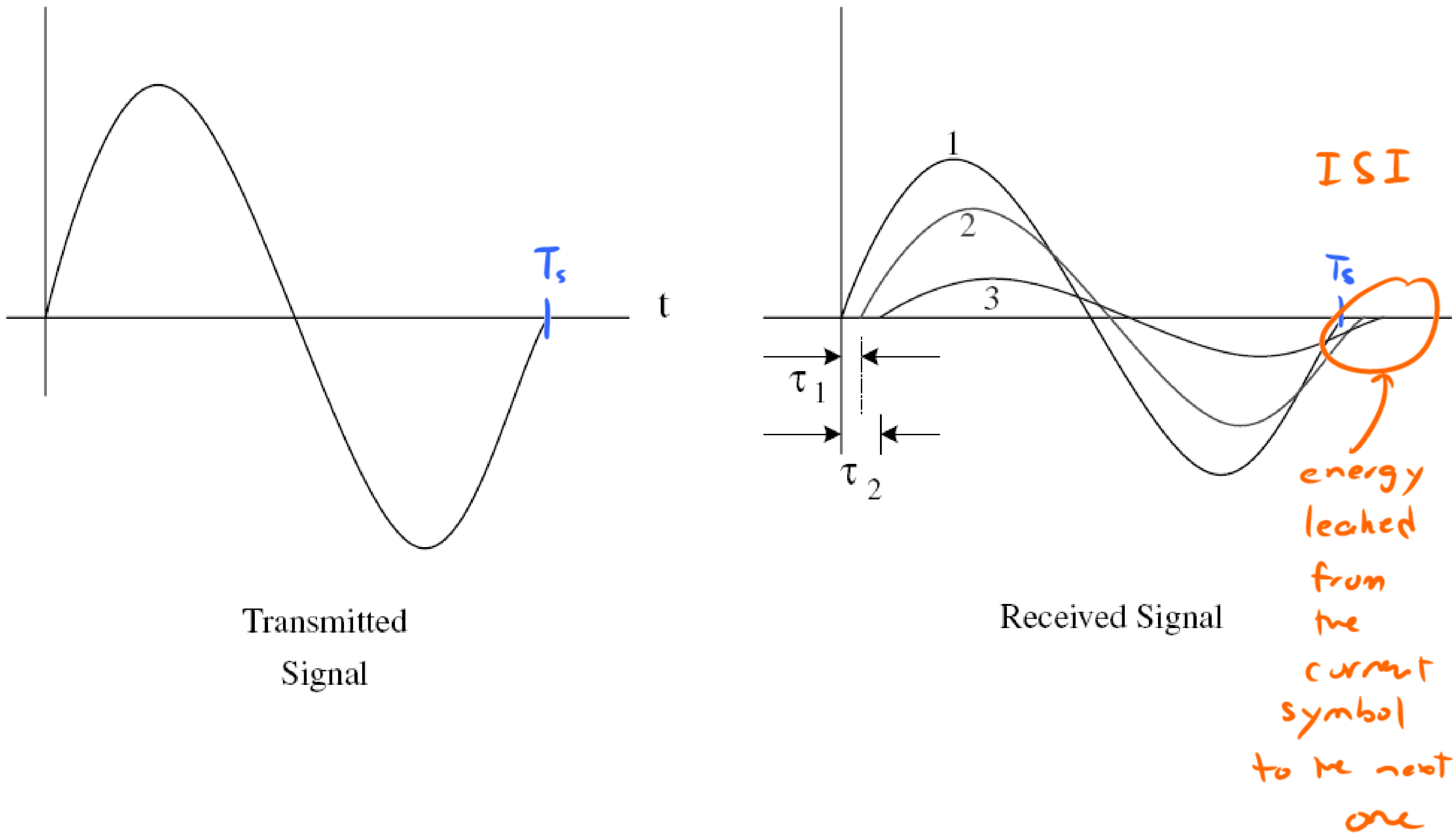
Three steps towards modern OFDM

1. Mitigate Multipath (ISI): Decrease the rate of the original data stream via multicarrier modulation (FDM)
2. Gain Spectral Efficiency: Utilize orthogonality
3. Achieve Efficient Implementation: FFT and IFFT

- Extra step: Completely eliminate ISI and ICI
  - Cyclic prefix

Cyclic Prefix: Motivation (1)

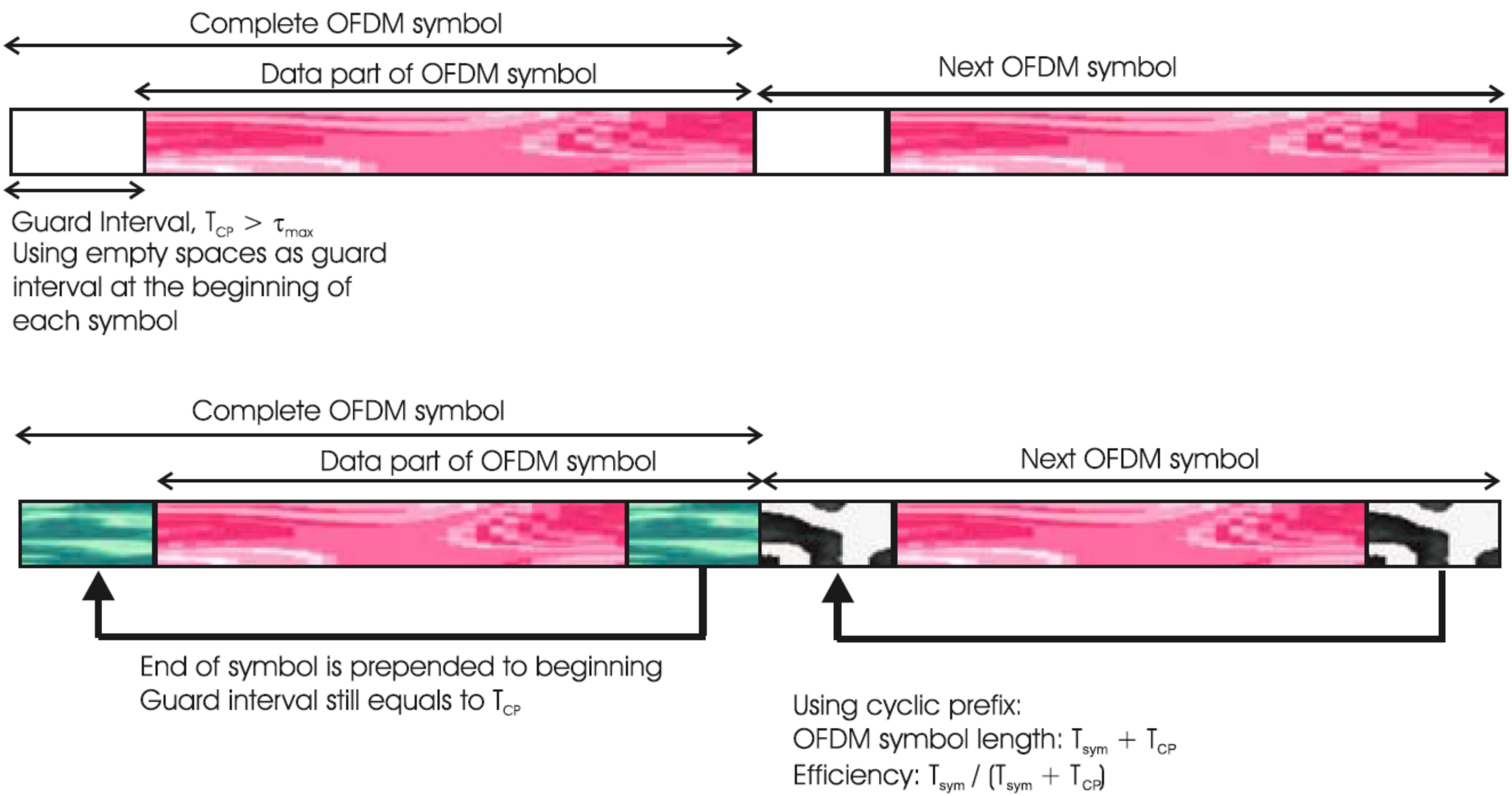
- Recall: Multipath Fading and Delay Spread



Cyclic Prefix: Motivation (2)

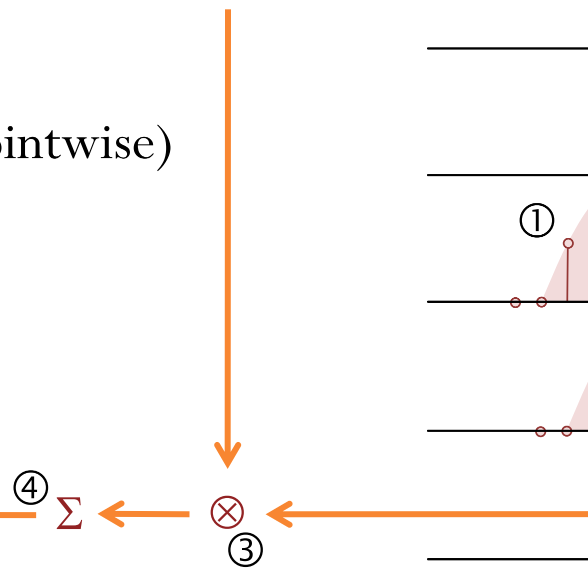
- OFDM uses large symbol duration T_s
  - compared to the duration of the impulse response τ_{max} of the channel
  - to reduce the amount of ISI
- Q: Can we "eliminate" the multipath (**ISI**) problem?
- To reduce the ISI, add **guard interval** larger than that of the estimated delay spread.
- If the guard interval is left empty, the orthogonality of the sub-carriers no longer holds, i.e., **ICI** (inter-channel interference) still exists.
- To prevent both the ISI as well as the ICI, OFDM symbol is **cyclically extended** into the guard interval.

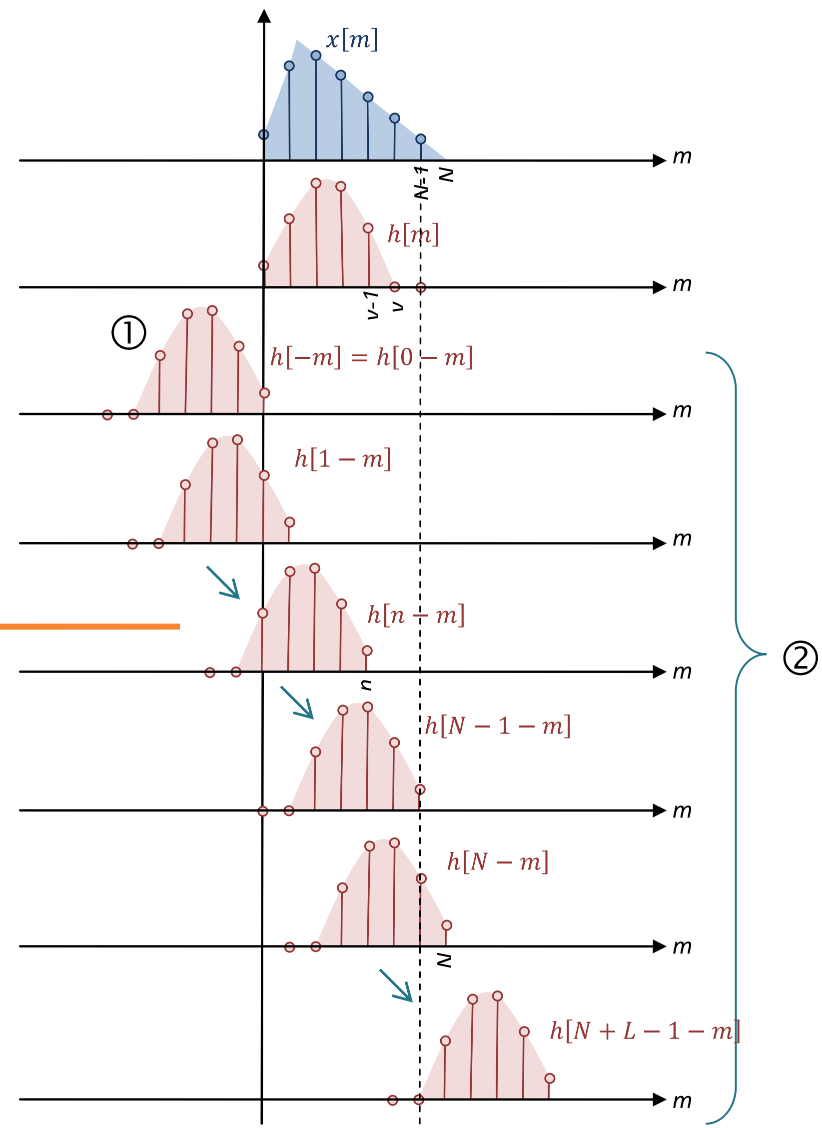
Cyclic Prefix



Recall: Convolution

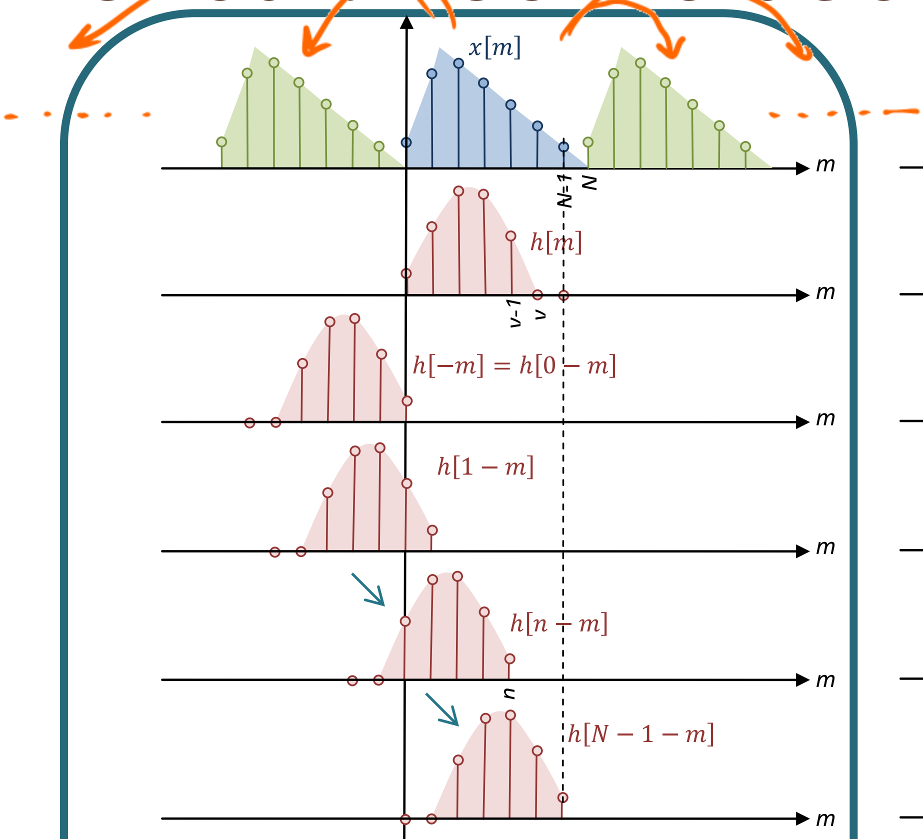
① Flip
② Shift
③ Multiply (pointwise)
④ Add





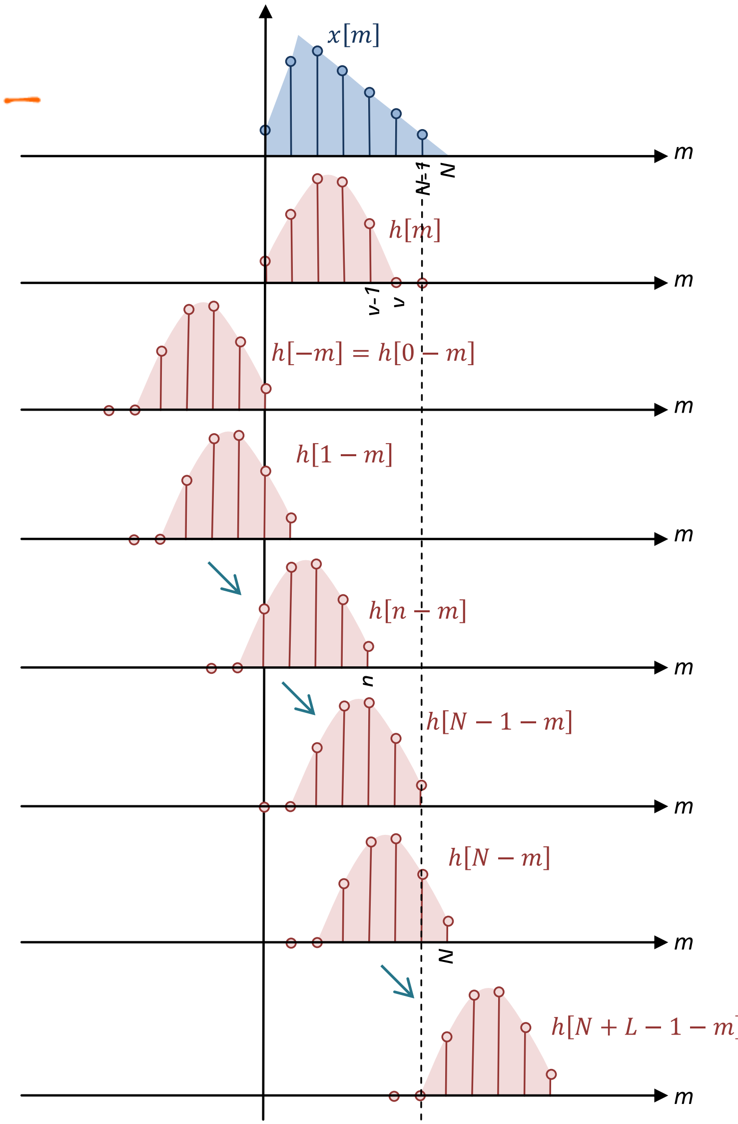
$$\{x * h\}[n] = \sum_{m} x[m]h[n-m]$$

Circular Convolution



Replicate *x* (now it looks periodic)
Then, perform the usual convolution
only on n = 0 to *N*-1

(Regular Convolution)



Circular Convolution: Examples 1

Find

$$[1 \quad 2 \quad 3] * [4 \quad 5 \quad 6]$$

$$[1 \quad 2 \quad 3] \circledast [4 \quad 5 \quad 6]$$

$$[1 \quad 2 \quad 3 \quad 0 \quad 0] \circledast [4 \quad 5 \quad 6 \quad 0 \quad 0]$$

Discussion

- Regular convolution of an N_1–point vector and an N_2–point vector gives (N_1+N_2-1)-point vector. 3 3 3+3-1 = 5
- Circular convolution is perform between two equal-length vectors. The results also has the same length.
- Circular convolution can be used to find the regular convolution by **zero-padding**.
  - Zero-pad the vectors so that their length is N_1+N_2-1.
  - Example:

$$[1 \quad 2 \quad 3 \quad 0 \quad 0] \circledast [4 \quad 5 \quad 6 \quad 0 \quad 0] = [1 \quad 2 \quad 3] * [4 \quad 5 \quad 6]$$

- In modern OFDM, we want to perform circular convolution via regular convolution.

Circular Convolution in Communication

- We want the receiver to obtain the circular convolution of the signal (channel input) and the channel.
- Q: Why?
- A:
  - **CTFT**: **convolution** in time domain corresponds to **multiplication** in frequency domain.
    - This fact does not hold for DFT.
  - **DFT**: circular **convolution** in (discrete) time domain corresponds to **multiplication** in (discrete) frequency domain.
    - We want to have multiplication in frequency domain.
    - So, we want circular convolution and not the regular convolution.
- Problem: Real channel does regular convolution.
- Solution: With **cyclic prefix**, regular convolution can be used to create circular convolution.

Example 2

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = ?$$

Let's look closer at how we carry out the circular convolution operation. Recall that we replicate the x and then perform the regular convolution (for N points)

Solution:

$$1 \quad -2 \quad 3 \quad 1 \quad 2 \quad 1 \quad -2 \quad 3 \quad 1 \quad 2 \quad 1 \quad -2 \quad 3 \quad 1 \quad 2 \ldots$$

0 0 1 2 3	$1\times1+2\times2+1\times3=1+4+3=8$
0 0 1 2 3	$2\times1+1\times2+(-2)\times3=2+2-6=-2$
0 0 1 2 3	$1\times1+(-2)\times2+3\times3=1-4+9=6$
0 0 1 2 3	$(-2)\times1+3\times2+1\times3=-2+6+3=7$
0 0 1 2 3	$3\times1+1\times2+2\times3=3+2+6=11$

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = [8 \quad -2 \quad 6 \quad 7 \quad 11]$$

Goal: Get these numbers using regular convolution

Example 2

Memory

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = ?$$

Observation: We don't need to replicate the x indefinitely. Furthermore, when h is shorter than x, we don't even need a full replica.

length here is the same as the memory length of the channel

Not needed in the calculation

1	−2	3	1	2	1	−2	3	1	2	1	−2	3	1	2

0 0 1 2 3 $\quad 1\times1+2\times2+1\times3=1+4+3=8$

0 0 1 2 3 $\quad 2\times1+1\times2+(-2)\times3=2+2-6=-2$

0 0 1 2 3 $\quad 1\times1+(-2)\times2+3\times3=1-4+9=6$

0 0 1 2 3 $\quad (-2)\times1+3\times2+1\times3=-2+6+3=7$

0 0 1 2 3 $\quad 3\times1+1\times2+2\times3=3+2+6=11$

$$[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0] = [8 \quad -2 \quad 6 \quad 7 \quad 11]$$

Example 2

Try this: use only the necessary part of the replica and then convolute (regular convolution) with the channel.

$$[1 \quad 2 \quad 1 \quad -2 \quad 3 \quad 1 \quad 2] * [3 \quad 2 \quad 1] = ?$$

Copy the **last ν samples** of the symbols at the **beginning** of the symbol. This partial replica is called the **cyclic prefix**.

$$1 \quad 2 \quad 1 \quad -2 \quad 3 \quad 1 \quad 2$$

1	2	3								$1\times 3 = 3$
	1	2	3							$1\times 2 + 2\times 3 = 2+6 = 8$
		1	2	3						$1\times 1 + 2\times 2 + 1\times 3 = 1+4+3 = 8$
			1	2	3					$2\times 1 + 1\times 2 + (-2)\times 3 = 2+2-6 = -2$
				1	2	3				$1\times 1 + (-2)\times 2 + 3\times 3 = 1-4+9 = 6$
					1	2	3			$(-2)\times 1 + 3\times 2 + 1\times 3 = -2+6+3 = 7$
						1	2	3		$3\times 1 + 1\times 2 + 2\times 3 = 3+2+6 = 11$
							1	2	3	$1\times 1 + 2\times 2 = 1+4 = 5$
								1	2 3	$2\times 1 = 2$

Junk!

Example 2

- We now know that

$$[1 \quad 2 \quad 1 \quad -2 \quad 3 \quad 1 \quad 2] * [3 \quad 2 \quad 1] = [3 \quad 8 \quad \underbrace{8 \quad -2 \quad 6 \quad 7 \quad 11}_{[1 \quad -2 \quad 3 \quad 1 \quad 2] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0]} \quad 5 \quad 2]$$

Cyclic Prefix

$x^{(1)}$

- Similarly, you may check that

$$[-2 \quad 1 \quad 2 \quad 1 \quad -3 \quad -2 \quad 1] * [3 \quad 2 \quad 1] = [-6 \quad -1 \quad \underbrace{6 \quad 8 \quad -5 \quad -11 \quad -4}_{[2 \quad 1 \quad -3 \quad -2 \quad 1] \circledast [3 \quad 2 \quad 1 \quad 0 \quad 0]} \quad 0 \quad 1]$$

Cyclic Prefix

$x^{(2)}$

Example 3

- We know, from Example 2, that

```
[ 1  2  1  -2   3   1  2] * [3  2  1] = [ 3   8  8  -2   6    7  11  5  2]
```

And that

```
[-2  1  2   1  -3  -2  1] * [3  2  1] = [-6  -1  6   8  -5  -11  -4  0  1]
```

- Check that

```
  [ 1  2  1  -2  3   1   2   0   0  0  0   0    0   0] * [3  2  1]
= [ 3  8  8  -2  6   7  11   5   2  0  0   0    0   0  0  0]
```

and

```
  [ 0  0  0   0  0   0   0  -2   1  2  1  -3   -2   1] * [3  2  1]
= [ 0  0  0   0  0   0   0  -6  -1  6  8  -5  -11  -4  0  1]
```

Example 4

- We know that

```
[ 1 2 1 -2  3  1 2] * [3 2 1] = [ 3  8 8 -2  6   7 11 5 2]
[-2 1 2  1 -3 -2 1] * [3 2 1] = [-6 -1 6  8 -5 -11 -4 0 1]
```

- Using Example 3, we have

```
    [ 1 2 1 -2 3 1  2 -2  1 2 1 -3  -2  1] * [3 2 1]
= ( [ 1 2 1 -2 3 1  2  0  0 0 0  0   0  0] )
   +[ 0 0 0  0 0 0  0 -2  1 2 1 -3  -2  1] )  * [3 2 1]
=   [ 3 8 8 -2 6 7 11  5  2 0 0  0   0  0 0 0]
   +[ 0 0 0  0 0 0  0 -6 -1 6 8 -5 -11 -4 0 1]
=   [ 3 8 8 -2 6 7 11 -1  1 6 8 -5 -11 -4 0 1]
```

Putting results together…

- Suppose $\underline{x}^{(1)}$ = [1 -2 3 1 2] and $\underline{x}^{(2)}$ = [2 1 -3 -2 1]
- Suppose $\underline{h}$ = [3 2 1]
- At the receiver, we want to get
  - [1 -2 3 1 2] $\circledast$ [3 2 1 0 0] = [8 -2 6 7 11]
  - [2 1 -3 -2 1] $\circledast$ [3 2 1 0 0] = [6 8 -5 -11 -4]
- We transmit [1 2 1 -2 3 1 2 -2 1 2 1 -3 -2 1].

  Cyclic prefix (1 2) Cyclic prefix (-2 1)

- At the receiver, we get

  [1 2 1 -2 3 1 2 -2 1 2 1 -3 -2 1] * [3 2 1]

  = [3 8 8 -2 6 7 11 -1 1 6 8 -5 -11 -4 0 1]

Junk! To be thrown away by the receiver. (3 8; -1 1; 0 1)

Circular Convolution: Key Properties

- Consider an N-point signal $x[n]$
- **Cyclic Prefix (CP) insertion**: If $x[n]$ is extended by copying the last ν samples of the symbols at the beginning of the symbol:

$$\hat{x}[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ x[n+N], & -v \le n \le -1 \end{cases}$$

- Key Property 1:

$$\{h \circledast x\}[n] = (h * \hat{x})[n] \text{ for } 0 \le n \le N-1$$

- Key Property 2:

$$\{h \circledast x\}[n] \xrightarrow{\text{FFT}} H_k X_k$$

OFDM with CP for Channel w/ Memory

memory length is assumed to be $= \nu$

- We want to send N samples $S_0, S_1, \ldots, S_{N-1}$ across noisy channel with memory.

(IDFT)

- First apply IFFT: $S_k \xrightarrow{\text{IFFT}} s[n]$
- Then, add cyclic prefix

$$\hat{s} = \left[s[N-\nu], \ldots, s[N-1], s[0], \ldots, s[N-1] \right]$$

- This is inputted to the channel.
- The output is

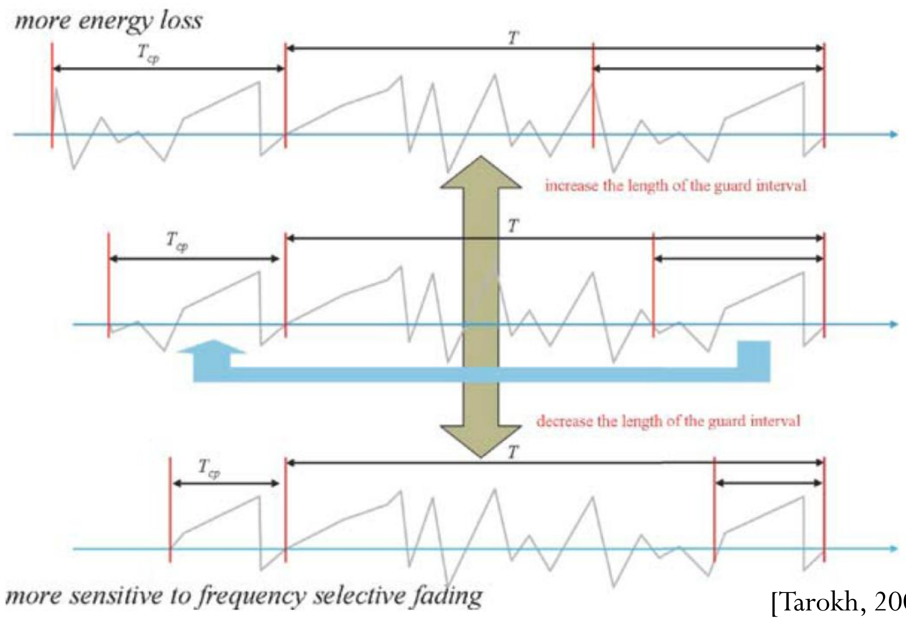
$$y[n] = \left[p[N-\nu], \ldots, p[N-1], r[0], \ldots, r[N-1] \right]$$

- Remove cyclic prefix to get $r[n] = h[n] \circledast s[n] + w[n]$
- Then apply FFT: $r[n] \xrightarrow{\text{FFT}} R_k$
- By circular convolution property of DFT, $R_k = H_k S_k + W_k$

No ICI!

OFDM System Design: CP

- A good ratio between the CP interval and symbol duration should be found, so that all multipaths are resolved and not significant amount of energy is lost due to CP.
- As a thumb rule, the CP interval must be two to four times larger than the root mean square (RMS) delay spread.



[Tarokh, 2009, Fig 2.9]

Reference

- A. Bahai, B. R. Saltzberg, and M. Ergen, ***Multi-Carrier Digital Communications: Theory and Applications of OFDM***, 2nd ed., New York: Springer Verlag, 2004.